MATH 2060 TUT04

<u>Def</u> 1) A fcn $f : [a,b] \to \mathbb{R}$ is said to be Riemann integrable on $[a,b]$ if $\exists\, L \in \mathbb{R}$ s.t. $\forall\, \varepsilon > 0$, $\exists\, \delta_\varepsilon > 0$ s.t.
$\forall$ tagged partition $\dot{P}$ of $[a,b]$ with $\|\dot{P}\| < \delta_\varepsilon$

$$|S(f;\dot{P}) - L| < \varepsilon$$

For $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$, $\quad \|\dot{P}\| = \max\{|x_i - x_{i-1}| : i = 1, \dots, n\}$

$$S(f;\dot{P}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$$

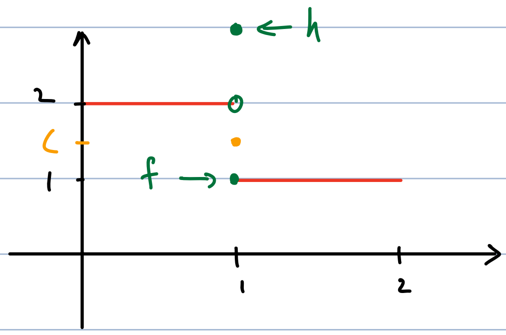
2) $\mathcal{R}[a,b] :=$ set of all Riemann integrable fcns on $[a,b]$

3) The number L is uniquely determined and is denoted by

$$\int_a^b f \qquad \text{or} \qquad \int_a^b f(x)\,dx$$

6. (a) Let $f(x) := 2$ if $0 \le x < 1$ and $f(x) := 1$ if $1 \le x \le 2$. Show that $f \in \mathcal{R}[0,2]$ and evaluate its integral.

(b) Let $h(x) := 2$ if $0 \le x < 1$, $h(1) := 3$ and $h(x) := 1$ if $1 < x \le 2$. Show that $h \in \mathcal{R}[0,2]$ and evaluate its integral.

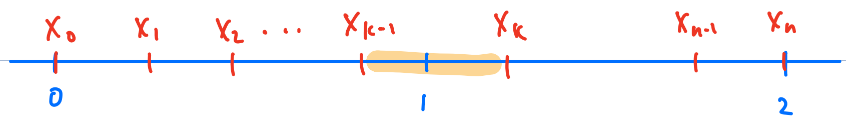


Ans: Fix $c \in \mathbb{R}$ and define $g : [0,2] \to \mathbb{R}$ by

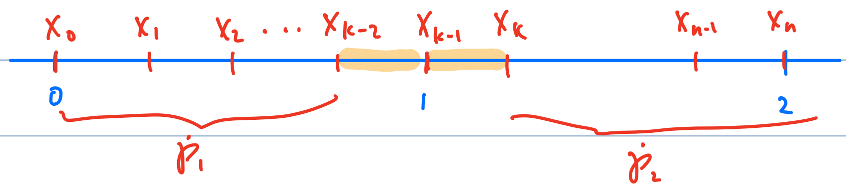
$$g(x) = \begin{cases} 2 & \text{if } 0 \le x < 1 \\ c & \text{if } x = 1 \\ 1 & \text{if } 1 < x \le 2 \end{cases}.$$

We will show that, regardless of the value of c, we always have $g \in \mathcal{R}[0,2]$ and $\int_0^2 g = 3$.

Let $\dot{P} := \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ be a tagged partition of $[0,2]$.
Suppose $x_{k-1} \le 1 < x_k$



or



Let $\dot{P}_1 := \{[x_{i-1}, x_i], t_i\}_{i=1}^{k-2}$, $\dot{P}_2 := \{[x_{i-1}, x_i], t_i\}_{i=k+1}^{n}$

Then

$$S(g;\dot{P}) = S(g;\dot{P}_1) + g(t_{k-1})(x_{k-1} - x_{k-2}) + g(t_k)(x_k - x_{k-1}) + S(g;\dot{P}_2),$$

where $S(g;\dot{P}_1) = \sum_{i=1}^{k-2} \cancel{g(t_i)}^{\,2}(x_i - x_{i-1}) = 2(x_{k-2} - x_0)$

$= 2 - 2(x_{k-1} - x_{k-2}) - 2(1 - x_{k-1})$

$S(g;\dot{P}_2) = \sum_{i=k+1}^{n} \cancel{g(t_i)}^{\,1}(x_i - x_{i-1}) = (x_n - x_k)$

$= 1 - (x_k - 1)$

Let $M = \max\{1, 2, |c|\}$. Then

$$|S(g;\dot{P}) - 3| \leq 2|x_{k-1} - x_{k-2}| + 2|1 - x_{k-1}| + |x_k - 1|$$

$$|g(t_{k-1})||x_{k-1} - x_{k-2}| + |g(t_k)||x_k - x_{k-1}|$$

(note $x_{k-1} \leq 1 < x_k$) $\quad \leq 2\|\dot{P}\| + 2\|\dot{P}\| + \|\dot{P}\| + 2M\|\dot{P}\|$

$$= (5 + 2M)\|\dot{P}\| .$$

Now, $\forall \varepsilon > 0$, take $\delta := \frac{\varepsilon}{5+2M} > 0$, so that any tagged partition $\dot{P}$ of $[0,2]$ with $\|\dot{P}\| < \delta$ satisfies

$$|S(g;\dot{P}) - 3| \leq (5+2M)\|\dot{P}\| < (5+2M)\delta = \varepsilon$$

Therefore $g \in \mathcal{R}[0,2]$ and $\int_0^2 g = 3$ //

8. If $f \in \mathcal{R}[a,b]$ and $|f(x)| \leq M$ for all $x \in [a,b]$, show that $\left|\int_a^b f\right| \leq M(b-a)$.

Ans: For any tagged partition $\dot{P} := \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ of $[a,b]$, we have

$$\sum_{i=1}^n (-M)(x_i - x_{i-1}) \leq S(f;\dot{P}) = \sum_{i=1}^n \underbrace{f(t_i)}_{-M \leq \ \leq M}\underbrace{(x_i - x_{i-1})}_{\geq 0} \leq \sum_{i=1}^n M(x_i - x_{i-1})$$

$$\Rightarrow \quad -M(b-a) \leq S(f;\dot{P}) \leq M(b-a)$$

$$\Rightarrow \quad |S(f;\dot{P})| \leq M(b-a)$$

Let $\varepsilon > 0$. Since $f \in \mathcal{R}[a,b]$, $\exists \delta > 0$ s.t. if $\dot{P}$ is a tagged partition of $[a,b]$ with $\|\dot{P}\| < \delta$, then

$$\left| S(f;\dot{P}) - \int_a^b f \right| < \varepsilon$$

Let $\dot{Q}$ be such a tagged partition. Then

$$\left|\int_a^b f\right| \leq \left|\int_a^b f - S(f;\dot{P})\right| + |S(f;\dot{P})|$$

$$\leq M(b-a) + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\left|\int_a^b f\right| \leq M(b-a)$$

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10. Let $g(x) := 0$ if $x \in [0,1]$ is rational and $g(x) := 1/x$ if $x \in [0,1]$ is irrational. Explain why $g \notin \mathcal{R}[0,1]$. However, show that there exists a sequence $(\dot{\mathcal{P}}_n)$ of tagged partitions of $[a,b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ and $\lim_n S(g;\dot{\mathcal{P}}_n)$ exists.

Ans: Let $P = \{[x_{i-1}, x_i]\}_{i=1}^{n}$ be a partition of $[0,1]$.

If we choose a rational tag r_i for each $[x_{i-1}, x_i]$,

then $S(g; \{[x_{i-1}, x_i], r_i\}_{i=1}^{n}) = \sum_{i=1}^{n} g(r_i)(x_i - x_{i-1}) = 0$

If we choose an irrational tag q_i for each $[x_{i-1}, x_i]$,

then $S(g; \{[x_{i-1}, x_i], q_i\}_{i=1}^{n}) = \sum_{i=1}^{n} g(q_i)(x_i - x_{i-1}) \geq 1$

We have for any $L \in \mathbb{R}$, $\exists\ \varepsilon_0 := 1/2 > 0$ s.t

$\forall \delta > 0$, $\exists$ a tagged partition $\dot{P}$ of $[0,1]$ s.t $\|\dot{P}\| < \delta$

$|S(g; \dot{P}) - L| \geq \varepsilon_0$

Therefore $g \notin R[0,1]$.

Finally, define $\dot{P}_n := \{[\frac{i-1}{n}, \frac{i}{n}], \frac{i}{n}\}_{i=1}^{n}$

Then $\|\dot{P}_n\| = \frac{1}{n} \to 0$

and $S(g; \dot{P}_n) = 0 \quad \forall n \in \mathbb{N}$

$\Rightarrow \quad \lim_n S(g; \dot{P}_n) = 0$

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12. Consider the Dirichlet function, introduced in Example 5.1.6(g), defined by $f(x) := 1$ for $x \in [0,1]$ rational and $f(x) := 0$ for $x \in [0,1]$ irrational. Use the preceding exercise to show that f is *not* Riemann integrable on $[0,1]$.

Recall: (Q11)

Suppose f is bounded on $[a,b]$ and that there exist two seqs of tagged partitions of $[a,b]$ s.t. $\|\dot{P}_n\| \to 0$, $\|\dot{Q}_n\| \to 0$, but s.t. $\lim_n S(f; \dot{P}_n) \neq \lim_n S(f; \dot{Q}_n)$.

Then $f \notin \mathcal{R}[a,b]$.

Ans: Let $(\dot{P}_n)$, $(\dot{Q}_n)$ be tagged partitions defined by

$$\dot{P}_n = \left\{ \left[\tfrac{i-1}{n}, \tfrac{i}{n}\right], \tfrac{i-1}{n} \right\}_{i=1}^{n}, \quad \dot{Q}_n = \left\{ \left[\tfrac{i-1}{n}, \tfrac{i}{n}\right], \tfrac{i-1}{n} + \tfrac{1}{n\sqrt{2}} \right\}_{i=1}^{n}$$

Then $\|\dot{P}_n\| = \|\dot{Q}_n\| = \frac{1}{n} \to 0$

However, $S(f; \dot{P}_n) = 1 \quad \forall n$

$S(f; \dot{Q}_n) = 0$

$\Rightarrow \quad \lim_n S(f; \dot{P}_n) = 1 \neq 0 = \lim_n S(f; \dot{Q}_n)$

Hence $f \notin \mathcal{R}[0,1]$ //

15. If $f \in \mathcal{R}[a,b]$ and $c \in \mathbb{R}$, we define g on $[a+c, b+c]$ by $g(y) := f(y-c)$. Prove that $g \in \mathcal{R}[a+c, b+c]$ and that $\int_{a+c}^{b+c} g = \int_a^b f$. The function g is called the c-**translate** of f.

Ans! If $\dot{P} := \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ is a tagged partition of $[a+c, b+c]$, define a tagged partition of $[a,b]$ by

$$\dot{P}_c := \{[x_{i-1}-c, x_i - c], t_i - c\}_{i=1}^n .$$

Clearly $\|\dot{P}_c\| = \|\dot{P}\|$.

Moreover,

$$S(g;\dot{P}) = \sum_{i=1}^n g(t_i)(x_i - x_{i-1}) = \sum_{i=1}^n f(t_i - c)\big((x_i - c) - (x_{i-1}-c)\big) = S(f;\dot{P}_c)$$

Let $\varepsilon > 0$. Since $f \in \mathcal{R}[a,b]$, $\exists \delta > 0$ s.t. if $\dot{Q}$ is any tagged partition of $[a,b]$ with $\|\dot{Q}\| < \delta$, then

$$\left| S(f;\dot{Q}) - \int_a^b f \right| < \varepsilon$$

Now, if $\dot{P}$ is a tagged partition of $[a+c, b+c]$ with $\|\dot{P}\| < \delta$, then $\dot{P}_c$ is a tagged partition of $[a,b]$ with $\|\dot{P}_c\| = \|\dot{P}\| < \delta$.

Hence $$\left| S(g;\dot{P}_c) - \int_a^b f \right| = \left| S(f;\dot{P}) - \int_a^b f \right| < \varepsilon .$$

Therefore $g \in \mathcal{R}[a+c, b+c]$ and $\int_{a+c}^{b+c} g = \int_a^b f$ //